

Determination of Landau Fermi-liquid parameters of strongly interacting fermions by means of a nonlinear scaling transformation

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A nonlinear transformation approach is formulated for the correlated fermions' thermodynamics through a medium-scaling effective action. An auxiliary implicit variable-effective chemical potential is introduced to characterize the non-Gaussian fluctuations physics. By incorporating the nonlocal correlation effects, the achieved grand partition function is made of coupled highly nonlinear parametric equations. Analytically, the low temperature expansions for the strongly interacting unitary Fermi gas are performed for the adiabatic compressibility-sound speed and specific heat with the Sommerfeld lemma. The expressions for the Landau Fermi-liquid parameters F_0^s and F_1^s of the strongly interacting fermion system are obtained. As a universal constant, the effective fermion mass ratio is $m^*/m = \frac{10}{9}$ at unitarity.

Keywords: Rigorous results in statistical mechanics, Bose Einstein condensation (Theory), Series expansions

The Landau theory of Fermi liquid plays the key role in understanding the novel interacting fermions quantum many-body physics[1]. Either in the non-relativistic or relativistic frameworks, the central task is how to calculate the Landau Fermi-liquid parameters[2, 3, 4, 5], which contributes to addressing the bulk or kinetic transport properties in terms of the quasi-particle viewpoint. The conventional techniques for calculating the effective fermion mass and/or ground state energy will involve the various coupled integral equations for the multi-points correlation Green functions exhibited by the Galitskii, Bethe-Sepeter or Bethe-Goldstone expansions[6].

In recent years, considerable efforts for understanding the physics from Bardeen-Cooper-Schrieffer to Bose-Einstein condensation(BCS-BEC crossover) with ultra-cold atomic Fermi gases have been made. The two-body interaction strength can be tuned with the Feshbach resonance. At unitarity, the divergent scattering length with a zero energy bound state can manifest the universal properties[7, 8, 9, 10].

The fundamental issue on the unitary fermions thermodynamics is on the symmetric fermions ground state energy[7, 8]. Another exciting one concerns the asymmetric system thermodynamics. To describe or interpret the phase separation properties achieved by the experimentally trapped systems, the Landau effective fermion mass becomes the key dynamical parameter[11, 12, 13]. Like in the ground state energy, the infinite scattering length will certainly drop out in the effective mass expression, which excites considerable attempts[11, 12, 14, 15]. The existing values are in the regime $m^*/m \sim 1.04 - 2.5$. What is the exact value of m^*/m ?

In this Letter, we will develop a non-perturbative statistical mechanical method to understand the novel strongly correlating physics. The comprehensive thermodynamical quantities can be fixed by the underlying

partition function. For instance, the entropy incorporating the dynamical and quantum correlations can be derived from the grand thermodynamical potential. As we will see, the entropy reasonably characterizing the strong correlation information is crucial in determining the Landau parameters while ensuring the strict thermodynamical self-consistency.

The basic prospective is that the single particle spectrum will be modified by the collective dispersive effects. The medium-dependent effective action allows a natural implementation of the collective effects as discussed in nuclear many-body literature[4, 5]. One can imagine that the bare interaction potential is renormalized by the surrounding environment and the single particle spectrum will be modified. At unitarity, the medium dependence of the interaction strength can cancel the infrared divergences and lead to a finite physical result. Fixing the Landau Fermi-liquid parameters of strongly interacting fermions at unitarity is the motivation.

In order to account for the unusual fluctuation/correlation effects, we work with the medium-scaling Hamiltonian in terms of the grand canonical ensemble framework[16]

$$\tilde{H} = - \int d^3x \psi_\alpha^*(x) \left(\frac{\nabla^2}{2m} - \mu_r[n, T] \right) \psi_\alpha(x) + \frac{U_{\text{eff}}^*[n, T]}{2} \int d^3x \psi_\alpha^*(x) \psi_\beta^*(x) \psi_\beta(x) \psi_\alpha(x). \quad (1)$$

In equation (1), $\alpha, \beta = \uparrow (a), \downarrow (b)$ represent the hyperfine-spin projection Ising-variables while the m is the bare fermion mass. The bare coupling constant $U_0 = 4\pi a/m$ has been substituted by a medium dependent functional through a specific transformation

$$U_{\text{eff}}^*[n, T] = \frac{U_0}{1 - \frac{1}{2}m_D^2 U_0}, \quad m_D^2 = \left(\frac{\partial n}{\partial \mu^*} \right)_T. \quad (2)$$

The n is the particle number density and μ^* the effective chemical potential defined below through equation (4).

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The *minus* sign in the denominator of equation (2) incorporates the alternating frustration function of the surrounding environment. The frustration spirit is consistent with the general Le Chatelier's stability principle in thermodynamics[16, 17]. The nonlocal correlation physics characterized by the alternating minus sign coincides with the second law of thermodynamics. Furthermore, the many-body correlation effects are taken into account as a spontaneously generated *single-body* potential $\propto \mu_r[n, T]$. Sticking to the medium dependence of U_{eff}^* , the additional compensatory term $\delta\mathcal{H} \propto \mu_r[n, T]N_\alpha$ can enforce the energy-momentum conservation law.

Two steps will be taken to give the grand thermodynamical potential with equation (1) and the medium-scaling interaction, equation (2). Firstly, the *shifted* relative minimum $\tilde{\Omega}(T, \tilde{\mu})$ is obtained by fixing the interaction analogous to the Hartree approximation with the linear bare potential. Secondly, the absolute minimum $\Omega(T, \mu)$ for the given chemical potential μ is derived with the constraint Legendre transformation correspondence relations between μ and N or T and S [16, 18].

The achieved equation of state/grand thermodynamical potential can be presented as the coupled *parametric equations* formalism

$$P = P_{\text{ideal}}(T, \mu^*) + \frac{\pi a_{\text{eff}}}{m} n^2 + \mathcal{C} \left(\frac{2\pi a_{\text{eff}}}{m} \right)^2 n^3, \quad (3)$$

$$\mu = \mu^* + \frac{2\pi a_{\text{eff}}}{m} n + \mathcal{C} \left(\frac{2\pi a_{\text{eff}}}{m} \right)^2 n^2, \quad (4)$$

with

$$P_{\text{ideal}} = \frac{2T}{\lambda^3} f_{5/2}(z'), \quad (5)$$

being a function of the effective chemical potential. The employed notation a_{eff} is defined by $U_{\text{eff}}^* \equiv 4\pi a_{\text{eff}}/m$.

With the implicit collective variable μ^* introduced by the single particle Green function self-consistent equation (4), the total number density $n = n_\uparrow + n_\downarrow = 2n_\uparrow$ is

$$n(T, \mu^*) \equiv 2 \int_k f_k = \frac{2}{\lambda^3} f_{3/2}(z'), \quad (6)$$

with the momentum integral symbol $\int_k = \int d^3\mathbf{k}/(2\pi)^3$ and quasi-particle Fermi-Dirac distribution functions defined as

$$f_k = \frac{1}{z'^{-1} e^{\beta \frac{\mathbf{k}^2}{2m}} + 1}, \quad z' = e^{\beta \mu^*}. \quad (7)$$

The $\lambda = \sqrt{2\pi/(mT)}$ is the mean thermodynamical de Broglie wavelength. The β is the inverse temperature (with $k_B = \hbar = 1$ throughout the letter). The *effective fugacity* z' is analogous to fugacity $z = e^{\beta \mu}$.

The first two terms of equation (3) have exactly the same structure of the mean-field theory with fixed local interaction, which is linear-like. The last correction term $\propto n^3$ is picked up in a thermodynamical way by relaxing the medium-dependence of interaction potential;

it describes the non-Gaussian correlation physics and of non-local characteristic. They are the *rearrangement* effects of single particle energy spectrum shift reflected by the Brueckner-Bethe-Goldstone (BBG) technique with medium-dependent interaction potential[5]. From the statistical field theory viewpoint, this term guarantees the energy-momentum conservation law.

Associated with $U_{\text{eff}}^*[n, T]$, the shift strength \mathcal{C} can be identified to be

$$\begin{aligned} \mathcal{C}(T, \mu^*) &= \frac{1}{2} \left(\frac{\partial m_D^2}{\partial n} \right)_T \\ &= \frac{1}{2T} \frac{f_{-1/2}(z')}{f_{1/2}(z')}. \end{aligned} \quad (8)$$

Furthermore, the entropy density $s = S/V$ can be derived with the Hamiltonian-Jacobi *implicit variable* method from the coupled equations, (3) and (4)

$$\begin{aligned} s &= \left(\frac{\partial P}{\partial T} \right)_\mu \\ &= s_{\text{ideal}} + \mathcal{D} \left(\frac{2\pi a_{\text{eff}}}{m} \right)^2 n^2, \end{aligned} \quad (9)$$

$$s_{\text{ideal}} \equiv -2 \int_k [f_k \ln f_k + (1 - f_k) \ln(1 - f_k)], \quad (10)$$

$$\mathcal{D}(T, \mu^*) = -\frac{1}{2} \left(\frac{\partial m_D^2}{\partial T} \right)_n. \quad (11)$$

There is a dual relation between \mathcal{C} and \mathcal{D} , which can be seen from equations (8) and (11). It is worthy noting that the correlation factor $\mathcal{D}(T, \mu^*)$ explicitly characterizes the temperature fluctuation in addition to density fluctuation. For the non-relativistic scenario, the factor \mathcal{D} can be furthermore simplified

$$\mathcal{D}(T, \mu^*) = -\frac{m_D^2}{4T} + \frac{3\mathcal{C}}{2T} n. \quad (12)$$

Correspondingly, the entropy and energy density $\epsilon = E/V = -P + \mu n + Ts$ can be also reduced to the compact formalisms with the explicit $\propto n^2, n^3$ correction terms

$$s = s_{\text{ideal}} - \frac{m_D^2}{T} \left(\frac{\pi a_{\text{eff}}}{m} \right)^2 n^2 + \frac{3\mathcal{C}}{2T} \left(\frac{2\pi a_{\text{eff}}}{m} \right)^2 n^3 \quad (13)$$

$$\begin{aligned} \epsilon &= \epsilon_{\text{ideal}} + \frac{3}{2} \left(1 - \frac{1}{3(1 - \frac{2\pi m_D^2 a}{m})} \right) \frac{\pi a_{\text{eff}}}{m} n^2 \\ &\quad + \frac{3\mathcal{C}}{2} \left(\frac{2\pi a_{\text{eff}}}{m} \right)^2 n^3. \end{aligned} \quad (14)$$

With $P_{\text{ideal}} = \frac{2}{3}\epsilon_{\text{ideal}}$, the Virial Theorem $PV = \frac{2}{3}E$ of ideal gas is satisfied at unitarity[8], which is obvious by comparing equation (3) and equation (14). The vanishing $\propto n^2, n^3$ terms in the entropy density at $T \rightarrow 0$ ensures the third law of thermodynamics as expected.

The thermodynamical properties can be characterized by the coupled parametric equations (3) and (4) or equations (13) and (14). Relevant formulae can be expressed

explicitly in terms of $s = S/V$ etc. and according to T and the implicit variable μ^* .

For example, the sound speed squared is given by the adiabatic compressibility $\kappa_S = (\partial n / \partial P)_S$ according to

$$c^2 = \left(\frac{\partial P}{\partial mn} \right)_S = \frac{1}{m} \left[\left(\frac{\partial P}{\partial n} \right)_s + \frac{s}{n} \left(\frac{\partial P}{\partial s} \right)_n \right]. \quad (15)$$

The specific heat per particle at constant volume is

$$\frac{C_V}{N} = \frac{T}{n} \left(\frac{\partial s}{\partial T} \right)_n. \quad (16)$$

The specific heat C_P per particle at constant pressure is calculated according to

$$\frac{C_P}{N} = \frac{T}{n} \left[\left(\frac{\partial s}{\partial T} \right)_P - \frac{s}{n} \left(\frac{\partial n}{\partial T} \right)_P \right]. \quad (17)$$

The general expressions for c^2 , C_V , C_P etc. at finite temperature involve the high order susceptibilities described by such as $f_{-5/2}(z')$. They are quite lengthy and are omitted for brevity. Obviously, the important role has been played by the entropy incorporating the dynamical high order and quantum fluctuation/correlation effects.

Now we take the Sommerfeld lemma to do the low temperature expansion in order to give the Landau parameters. To characterize the strong correlation physics, the expansion is in terms of $T/\mu^* = (\ln z')^{-1}$ instead of directly according to $(\ln z)^{-1}$.

Let us start from the standard Fermi integral $f_{5/2}(z')$ [6]

$$f_{5/2}(z') = \frac{4}{3\sqrt{\pi}} I(\alpha), \quad \alpha = \frac{\mu^*}{T}; \quad I(\alpha) \equiv \int_{-\alpha}^{\infty} \frac{(x + \alpha)^{3/2}}{e^x + 1} dx.$$

In the strong degenerate regime with $\mu^* \gg T$, $I(\alpha)$ and Fermi integral $f_{5/2}(z')$ can be approximated by

$$I(\alpha) = \frac{2\alpha^{5/2}}{5} + \frac{\pi^2 \alpha^{1/2}}{4} - \frac{7\pi^4}{960\alpha^{3/2}} + \dots; \quad (18)$$

$$f_{5/2}(z') = \frac{8\mu^{*5/2}}{15\sqrt{\pi}T^{5/2}} + \frac{\pi^{3/2}\mu^{*1/2}}{3T^{1/2}} - \frac{7\pi^{7/2}T^{3/2}}{720\mu^{*3/2}} + \dots \quad (19)$$

From equation (19), the low temperature expansions of the remaining Fermi integrals can be obtained by using

$$f_{j-1}(z') = z' \frac{\partial f_j(z')}{\partial z'} = T \frac{\partial f_j(z')}{\partial \mu^*}. \quad (20)$$

With

$$f_{3/2}(z') = \frac{4\mu^{*3/2}}{3\sqrt{\pi}T^{3/2}} + \frac{\pi^{3/2}T^{1/2}}{6\mu^{*1/2}} + \frac{7\pi^{7/2}T^{5/2}}{480\mu^{*5/2}} + \dots \quad (21)$$

and the particle number density $n = k_f^3/(3\pi^2)$, the effective chemical potential can be inversely expanded in terms of the ‘‘Fermi’’ momentum k_f

$$\mu^* = \frac{k_f^2}{2m} - \frac{\pi^2 m T^2}{6k_f^2} - \frac{\pi^4 m^3 T^4}{10k_f^6} + \dots \quad (22)$$

Inserting the intermediate result, equation (22), into the expansions such as equation (19) of the various Fermi integrals $f_j(z')$ ($j = -\frac{5}{2}, -\frac{3}{2}, \dots, \frac{5}{2}$) in C_V , c^2 and P etc., the thermodynamical quantities can be expanded according to T/T_f . Here, $T_f = k_f^2/(2m)$ is the ‘‘Fermi’’ characteristic temperature. The calculation is quite lengthy by not complicated and the explicit expressions can be obtained.

For example, the expansion for pressure is

$$P = \left(1 + \frac{5ak_f}{3(\pi - 2ak_f)} + \frac{10(ak_f)^2}{9(\pi - 2ak_f)^2} \right) \frac{k_f^5}{15m\pi^2} + \left(1 - \frac{8(ak_f)^3}{9(\pi - 2ak_f)^3} \right) \frac{k_f m T^2}{9} - \left(1 - \frac{176(ak_f)^2}{81(\pi - 2ak_f)^2} + \frac{236(ak_f)^3}{81(\pi - 2ak_f)^3} - \frac{40(ak_f)^4}{27(\pi - 2ak_f)^4} \right) \frac{\pi^2 m^3 T^4}{15k_f^3} + \dots, \quad (23)$$

where the factors not in the brackets are those for the ideal Fermi gas. By keeping up to the lowest order, equation (23) gives the universal coefficient $\xi = \frac{4}{9}$ to which much attention has been paid. At unitarity, the universal coefficient of the next order is $\frac{10}{9}$. It coincides with the first order coefficient of C_V/N , equation (28), when $|a| = \infty$. However, the general expressions are quite different from each other.

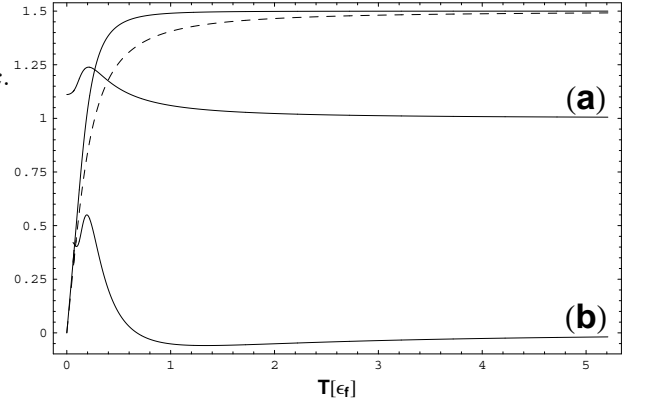


FIG. 1: Universal specific heat: The above solid curve is C_V/N of the unitary gas while the dashed one is C_V^0/N for ideal Fermi gas. The labeled (a) curve is the $C_V/C_V^{(0)}$ ratio while the indicated (b) is for $\partial^2 \mu / \partial n \partial T$.

The sound speed squared is expanded as

$$c^2 = \left(\frac{1 + F_0^s}{1 + \frac{1}{3}F_1^s} \right) \frac{k_f^2}{3m^2} + \left(1 - \frac{16ak_f}{5(\pi - 2ak_f)} - \frac{16(ak_f)^2}{5(\pi - 2ak_f)^2} - \frac{32(ak_f)^3}{45(\pi - 2ak_f)^3} + \frac{16(ak_f)^4}{45(\pi - 2ak_f)^4} + \frac{16(-10(ak_f)^2 + 9ak_f\pi)}{5(40(ak_f)^2 - 36ak_f\pi + 9\pi^2)} \right) \frac{5\pi^2 T^2}{9k_f^2} + \dots \quad (24)$$

The lengthy second expansion coefficient is still $\frac{10}{9}$ at unitarity. By comparing with those of Landau Fermi-liquid theory[2], the first expansion coefficient is

$$\frac{1 + F_0^s}{1 + \frac{1}{3}F_1^s} = 1 + \frac{2ak_f}{\pi - 2ak_f} + \frac{20(ak_f)^2}{9(\pi - 2ak_f)^2} + \frac{8(ak_f)^3}{9(\pi - 2ak_f)^3}, \quad (25)$$

The lowest order of the low temperature expansion for specific heat gives the effective mass definition

$$\frac{C_V}{N} = \left(\frac{m^*}{m}\right) \frac{\pi^2 m T}{k_f^2} - \left(1 - \frac{88(ak_f)^2}{27(\pi - 2ak_f)^2} + \frac{40(ak_f)^3}{27(\pi - 2ak_f)^3}\right) \frac{6\pi^4 (mT)^3}{5k_f^6} + \dots; \quad (26)$$

$$\frac{C_P}{N} = \left(\frac{m^*}{m}\right) \frac{\pi^2 m T}{k_f^2} + \left(1 - \frac{160ak_f}{3\pi - 4ak_f} + \frac{320ak_f}{9(\pi - 2ak_f)} + \frac{64(ak_f)^2}{9(\pi - 2ak_f)^2} - \frac{40(ak_f)^3}{9(\pi - 2ak_f)^3} - \frac{20(-8(ak_f)^2 + 3ak_f\pi)}{9(8(ak_f)^2 - 8ak_f\pi + 3\pi^2)}\right) \frac{2\pi^4 (mT)^3}{15k_f^6} + \dots; \quad (27)$$

$$\frac{m^*}{m} = 1 + \frac{1}{3}F_1^s = 1 + \frac{4(ak_f)^2}{9(\pi - 2ak_f)^2}. \quad (28)$$

The explicit difference between C_V/N and C_P/N is from the second order. At unitarity with $|a| = \infty$, the effective mass ratio is $m^*/m = \frac{10}{9}$, which agrees with the previous attempts 1.04 – 1.17 [11, 12, 14]. The effective fermion mass is the dynamical “higher order” effect because the linear term $\propto k_f a$ is absent in the weak coupling expansion[6]. The reciprocal term $\propto 1/(k_f a)$ does appear in the strong interaction regime.

The universal property of unitary Fermi gas can be also explored by such as the specific heat at unitarity. The

scaling function C_V/C_V^0 for C_V to C_V^0 of ideal Fermi gas is $\frac{10}{9}$ at $T \rightarrow 0$; it must be saturated as $C_V/C_V^0 \rightarrow 1$ in the Boltzmann limit. It is not a naive monotonously decreasing function with the increase of the rescaled temperature T/T_f . The numerical study indicates that this ratio has a knee peak at $T^* \approx 0.21T_f$ although C_V/N itself is a monotonously increasing function (different from the finite interaction scenario). This can be attributed to the competition between the particle number and temperature fluctuations. To characterize this unusual behavior, the mixing susceptibility $\partial^2 \mu / \partial T \partial n$ is shown in Fig.1, where a similar knee peak around T^* appears.

To conclude, the medium scaling spirit is realized by a nonlinear transformation for the effective interaction and consequently for the physical chemical potential. The introduced auxiliary implicit variable-effective chemical potential characterizes the correlation physics and makes the exact grand partition function appear as the highly nonlinear coupled parametric-equations.

The low temperature expansion in terms of $(\ln z')^{-1}$ (consequently eliminating z') can be performed analytically with the Sommerfeld method. The effective fermion mass and sound speed squared have been explicitly calculated. As the important dynamical parameter in Landau Fermi-liquid theory, the calibrated effective mass is reasonably consistent with some Monte-Carlo attempts. The parametric-equations formulation can gauge the infrared singularity manifested by the low-energy long wavelength thermodynamics of the three-dimensional contact interaction fermions system.

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